Proving the Conjecture $(-1)^\omega = 0$

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1 Introduction

The limit

$$\lim_{\omega\to\infty} (-1)^\omega = 0 \ \text{ in } {}^*\mathbb{Z} \tag{1}$$

for the hyperinteger ω is of interest in the analysis of divergent series. Research by the present authors suggested a conjecture that this limit is zero (Bartlett, Gaastra, and Nemati, 2020). Here, we prove that limit.

2 Initial Considerations

Consider ω to be a hyperinteger. We will specify this as $\omega \in {}^*\mathbb{Z}$.

For any integer $k \in \mathbb{Z}$, we have $\sin(\pi k) = 0$. By the transfer principle, this also holds for any hyperinteger $\omega \in {}^*\mathbb{Z}$. In other words,

$$\sin(\pi\omega) = 0 \ \ \forall\ \omega \in {}^*\mathbb{Z}. \tag{2}$$

By Euler's formula, we can determine that

$$e^{i\pi\omega} = \cos(\pi\omega) + i\sin(\pi\omega). \tag{3}$$

Using (2), (3) reduces to

$$e^{i\pi\omega} = \cos(\pi\omega) \ \text{ in } {}^*\mathbb{Z}. \tag{4}$$

3 Series Analysis

Consider the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k},$$

which converges to $\cos x$. This means that for any $N \in \mathbb{Z}$, for the partial sum

$$S_N(x) = \sum_{k=0}^{N} \frac{(-1)^k}{(2k)!} x^{2k},$$

it holds that

$$\lim_{N\to\infty} S_N(x) = \cos x \ \text{ in } \mathbb{Z}.$$

Therefore, by the transfer principle, for the partial sum

$$S_\Omega(x) = \sum_{k=0}^{\Omega} \frac{(-1)^k}{(2k)!} x^{2k},$$

we have

$$\lim_{\Omega\to\infty} S_\Omega(x) = \cos x \ \text{ in } {}^*\mathbb{Z}. \tag{5}$$

Evaluating S_Ω, we get

$$S_\Omega(x) = \cos x + (-1)^\Omega f(\Omega, x) \ \ \forall\ \Omega \in {}^*\mathbb{Z}, \tag{6}$$

where

$$f(\Omega, x) = \frac{x^{2\Omega+2}}{\Gamma(3+2\Omega)}\ {}_pF_q\left(\{1\}, \left\{\frac{3}{2}+\Omega, 2+\Omega\right\}, -\frac{x^2}{4}\right),$$

in which Γ is the Euler Γ function, and ${}_pF_q$ is the generalized hypergeometric function.

Consider now S_Ω when $x = \pi\omega$ for $\omega \in {}^*\mathbb{Z}$. Since the limit (5) exists and is finite for any x, then passing to the limit in (6) as $\Omega \to \infty$, we get

$$\cos(\pi\omega) + \lim_{\Omega\to\infty} (-1)^\Omega f(\Omega, \pi\omega) = \cos(\pi\omega) \ \ \forall\ \omega \in {}^*\mathbb{Z},$$

implying that

$$\lim_{\Omega\to\infty} (-1)^\Omega f(\Omega, \pi\omega) = 0 \ \ \forall\ \omega \in {}^*\mathbb{Z}. \tag{7}$$

However, when ω increases in ${}^*\mathbb{Z}$, f diverges to infinity, leading (7) to hold for all $\omega \in {}^*\mathbb{Z}$ if and only if

$$\lim_{\Omega \to \infty} (-1)^{\Omega} = 0 \text{ in } {}^*\mathbb{Z}.$$

Thus, the conjecture (1) holds true.

References

Bartlett, J, L Gaastra, and D Nemati (2020). "Hyperreal Numbers for Infinite Divergent Series". In: *Communications of the Blyth Institute* 2.1, pp. 7–15. DOI: 10.33014/issn.2640-5652.2.1.bartlett-et-al.1.