The Products of Hyperreal Series and the Limitations of Cauchy Products

Jonathan Bartlett

January 9, 2021

Bartlett, Gaastra, and Nemati (2020) set out to define a new way of assigning values to divergent series using hyperreal numbers, which we will refer to as the BGN method. This method operates using a small number of principles to allow value assignment for a divergent series:

1. Instead of summing to the ambiguous ∞, summations are done to a specific standard candle of hyperreal infinity, labeled ω.

2. Because summations are done to a specific infinity, then making sure that the number of positions are maintained is important.

3. When these rules are kept, divergent series summation can be done just like partial sums up to a value of k, where $k = \omega$.

Because of 2, $1+2+3+\dots$ refers to a different value than $1+0+2+0+3+0+\dots$, because the latter only has numbers in half of the positions as the former. To get both of these series to sum to the same value, the former series would sum from 1 to ω, while the later series would have to sum from 1 to 2ω. Because of 3, rearrangements of the series do not affect the value of the series. Additionally, standard summation formulas and discrete integrals can be used to simplify results.

As an example, the series

$$A = \sum_{i=1}^{\omega} i \tag{1}$$

can be given a value using the standard arithmetic series formula, yielding a value of $\frac{\omega^2}{2} - \frac{\omega}{2}$. This can be rounded to the primary part of the result, $\frac{\omega^2}{2}$.

Essentially, using hyperreals allows one to treat divergent series as if they were finite series. The point of this article is to point out that, while this makes divergent series work as if they were finite series, it *does not* make divergent series work as if they were convergent series. This is an important distinction that is easy to miss.

Figure 1: Cauchy Multiplication of Six Terms

	a_1	a_2	a_3	a_4	a_5	a_6
b_1	a_1b_1	a_2b_1	a_3b_1	a_4b_1	a_5b_1	a_6b_1
b_2	a_1b_2	a_2b_2	a_3b_2	a_4b_2	a_5b_2	a_6b_2
b_3	a_1b_3	a_2b_3	a_3b_3	a_4b_3	a_5b_3	a_6b_3
b_4	a_1b_4	a_2b_4	a_3b_4	a_4b_4	a_5b_4	a_6b_4
b_5	a_1b_5	a_2b_5	a_3b_5	a_4b_5	a_5b_5	a_6b_5
b_6	a_1b_6	a_2b_6	a_3b_6	a_4b_6	a_5b_6	a_6b_6

1 The Cauchy Product

In this article, we will focus on the Cauchy product. The Cauchy product for two series A and B is

$$A \cdot B = \sum_{i=1}^{\omega} \left(\sum_{j=1}^{i} a_j \, b_{i-j+1} \right). \tag{2}$$

For convergent series, the Cauchy product is equivalent to the product of A and B. However, *even with hyperreals*, the Cauchy product is not equivalent to th product of A and B if A and B are divergent.

To understand why, it is important to recognize the shape of the Cauchy product for finite series. Imagine that A and B are finite series with k elements. In that case, it is easy to recognize that the sum does not work.

Let us imagine two series with six elements. The Cauchy product for such a series would be

$$A \cdot B = \sum_{i=1}^{6} \left(\sum_{j=1}^{i} a_j \, b_{i-j+1} \right). \tag{3}$$

Figure 1 shows what this would look like. Each stripe of the figure represents one iteration through the outermost summation. Notice, however, that there are no stripes past the center stripe. In other words, all of the values past the center stripe are *not considered* in the final summation.

This is obviously problematic for finite sums. Why is it non-problematic for convergent series?

For a convergent series, as the index approaches infinity, the value of the the term approaches zero. If you look at Figure 1, it is evident that the terms that are being ignored, if it is a convergent series, have at least one multiplicand of the term at a near-zero value. Therefore, in a convergent series, all of the terms being ignored are negligible.

However, that is *not* true of a finite series. When performing hyperreal summations, the values and manipulations are similar to *finite* series, not necessarily similar to *convergent* series.

Note that Merten's theorem can be visualized from Figure 1. If one of the series is absolutely convergent, it will drag the conditionally convergent series

Figure 2: The BGN Product of A and B

	a_1	a_2	a_3	a_4	a_5	a_6
b_1	a_1b_1	a_2b_1	a_3b_1	a_4b_1	a_5b_1	a_6b_1
b_2	a_1b_2	a_2b_2	a_3b_2	a_4b_2	a_5b_2	a_6b_2
b_3	a_1b_3	a_2b_3	a_3b_3	a_4b_3	a_5b_3	a_6b_3
b_4	a_1b_4	a_2b_4	a_3b_4	a_4b_4	a_5b_4	a_6b_4
b_5	a_1b_5	a_2b_5	a_3b_5	a_4b_5	a_5b_5	a_6b_5
b_6	a_1b_6	a_2b_6	a_3b_6	a_4b_6	a_5b_6	a_6b_6

to zero in the terms that are being ignored.

2 Non-Cauchy Products

An advantage of Cauchy products is that it defines a way of understanding the behavior of a series in terms of a non-trivial manipulation of the previous series. It is obvious that we could represent the multiplication of hyperreal series A and B as

$$A \cdot B = \left(\sum_{i=1}^{\omega} a_i \right) \cdot \left(\sum_{i=1}^{\omega} b_i \right) \tag{4}$$

but that gives little additional information.

Another way of considering the multiplication is to take each element of A and multiply it by each element of B. This can be expressed as

$$A \cdot B = \sum_{i=1}^{\omega} \left(\sum_{j=1}^{\omega} a_i \, b_j \right) . \tag{5}$$

This, however, is not as useful as it could be, since it basically goes through every element of B *before* considering even the second element of A. Therefore, if there is any amount of convergence, it will not assist in finding such a value here. Also, it will not give much information about the behavior of partial sums.

A more informative formula can be found by tracing paths as outlined in Figure 2, which we will call the BGN method. While the multiplication method for Cauchy products is triangular (thus removing half of the terms), the BGN method is rectangular. At each index of the outer sum, all of the indices in both A and B up to that index are considered.

This method can be represented by the formula

$$A \cdot B = \sum_{i=1}^{\omega} \left(\left(\sum_{j=1}^{i} a_i \, b_j \right) + \left(\sum_{k=1}^{i-1} a_k \, b_i \right) \right) . \tag{6}$$

Because this formula incorporates *every* term of the series A and B, then the result will be equal to the naive multiplication in (4). Additionally, to the extent

Figure 3: $(1 + 1 + 1 + \ldots) \cdot (1 + -1 + 0 + 0 + 0 + \ldots)$

	1	1	1	1	1	1
1	1	1	1	1	1	1
-1	-1	-1	-1	-1	-1	-1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

that the beginning of the series are representative of the series as a whole, this will allow partial sums of the series to be used as approximations of the series as a whole. It also has the same number of elements as both the existing series, so it can easily be used operations with other series of the same shape.

3 An Example

The multiplication that first sparked this consideration is the product of the series $1 + 1 + 1 + \ldots$ multiplied by the series $1 + -1 + 0 + 0 + 0 + \ldots$ (with zeroes continuing). When put in the form specified by (1), the former sequence is equivalent to ω. The latter sequence is obviously 0.

$\omega \cdot 0 = 0$, however, the Cauchy product of these two series actually turns out to be 1. The reason for this can be seen in Figure 3. As can be seen, everything is included in this product except one term: -1. Therefore, since the entire product is 0, the product without the final -1 term will be 1.

This Cauchy product has this shape no matter how many terms (even an infinite number of terms), and therefore will always have the result of 1. However, in both the finite case and the infinite case, the *actual* product of the two series is zero.

4 Additional Considerations

The considerations given here should demonstrate why Cauchy products can be useful for convergent series while not being useful at all for multiplying two divergent series. Additionally, it should be evident that, even if the formula for the Cauchy product of a divergent series converges, it does not mean that the Cauchy product represents in any significant way the true value of the product of the two series. It simply means that, when half of the terms are not considered, the result is convergent. That does not yield a significant amount of confidence in such a result.

5 Conclusion

In (Bartlett, Gaastra, and Nemati, 2020), Section 11.2 hedged on the consideration of series rearrangement, suggesting that we could not rule out that rearranging series might cause the series to differ by an infinitesimal. However, the problem was that we were considering the results of Cauchy Products, which, here, we have shown are not representative of true products of divergent series.

As already noted, divergent series can, using hyperreals, be evaluated in the same way as if they were finite series. However, even though they can be treated similar to finite series, that is not the same as saying that they can be treated as convergent series.

References

Bartlett, J, L Gaastra, and D Nemati (2020). "Hyperreal Numbers for Infinite Divergent Series". In: *Communications of the Blyth Institute* 2.1, pp. 7–15.